Solution E1 /version 3 (Important: In this document decimal comma is used instead of decimal point in graphs and tables)

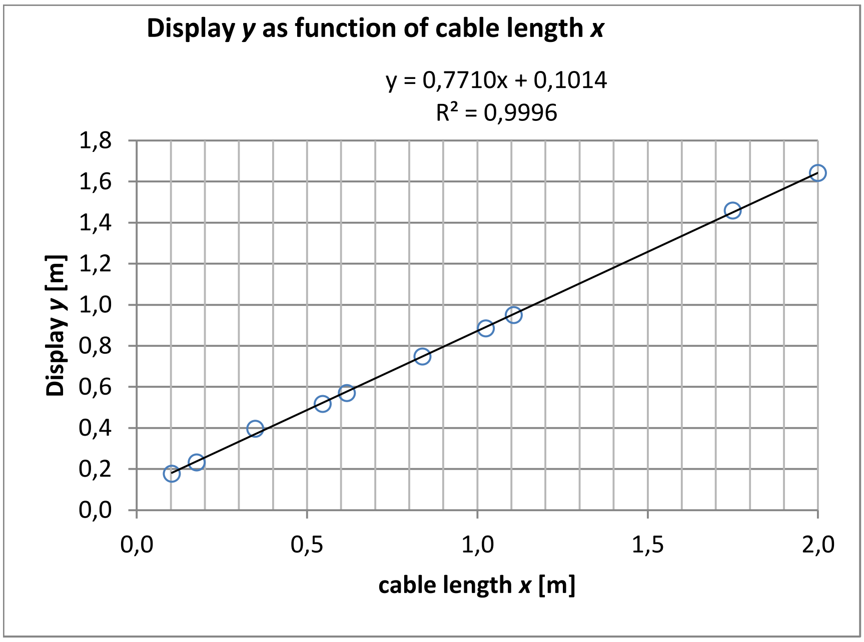
1.1

$H = 907\ \mathrm{mm}\ \pm 2\ \mathrm{mm}$. See the sketch in the figure corresponding to 1.3b. It must appear how the height is measured with the LDM in the rear mode.

1.2a

I used a 2 m cable but 1 m is sufficient. There should be about 8 lengths evenly distributed in the interval [0; 1 m].

x	*y*
m	m
0,103	0,177
0,176	0,232
0,348	0,396
0,546	0,517
0,617	0,570
0,839	0,748
1,025	0,885
1,107	0,950
1,750	1,459
2,000	1,642



1.2b

The refractive index is twice the gradient of the linear graph, $n_{\mathrm{co}} = 2 \cdot 0.7710 = 1.542$.

The reason for that is that the travel time for a light pulse

$$t = \frac{x}{v_{\mathrm{co}}} = \frac{x n_{\mathrm{co}}}{c}$$

The display will therefore show $y = \frac{1}{2}ct + k \Leftrightarrow y = \frac{1}{2}n_{\mathrm{co}}x + k.$

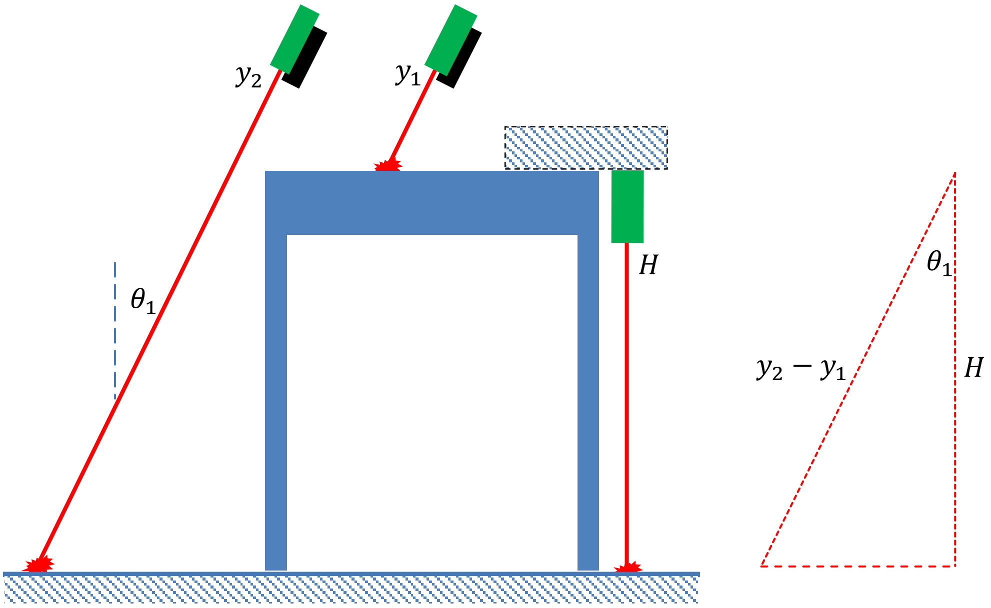
Lysets fart i lyslederkablet er $v_{\mathrm{co}} = \frac{c}{n_{\mathrm{co}}} = \frac{3{,}00 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}}}{1{,}542} = 1.95 \cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$

1.3a

$y_1 = 312\text{ mm} \pm 2\text{ mm}$, $y_2 = 1273\text{ mm} \pm 2\text{ mm}$

1.3b

$\theta_1 = \cos^{-1}\left(\frac{H}{y_2-y_1}\right) = \cos^{-1}\left(\frac{907\text{ mm}}{961\text{ mm}}\right) = 19.30°$, se figure:



Measuring the horizontal part of some triangle is very inaccurate because of the size of the laser dot. No marks will be awarded for that

Using $\delta = 2$ mm as the uncertainty of y_1, y_2 and H, one can calculate the uncertainty of θ_1

$$\Delta \cos\theta_1 = \Delta\left(\frac{H}{y_2 - y_1}\right)$$

Using simple derivative, we get

$$\sin\theta_1 \cdot \Delta\theta_1 = \frac{\delta}{H} + \frac{2\delta}{y_2 - y_1}$$

$$\Delta\theta_1 = \frac{\left(\frac{\delta}{H} + \frac{2\delta}{y_2 - y_1}\right)}{\sin\theta_1} \cdot \frac{180°}{\pi} = \frac{\left(\frac{2}{907} + \frac{4}{961}\right)}{\sin 19{,}30°} \cdot \frac{180°}{\pi} = 1.1°$$

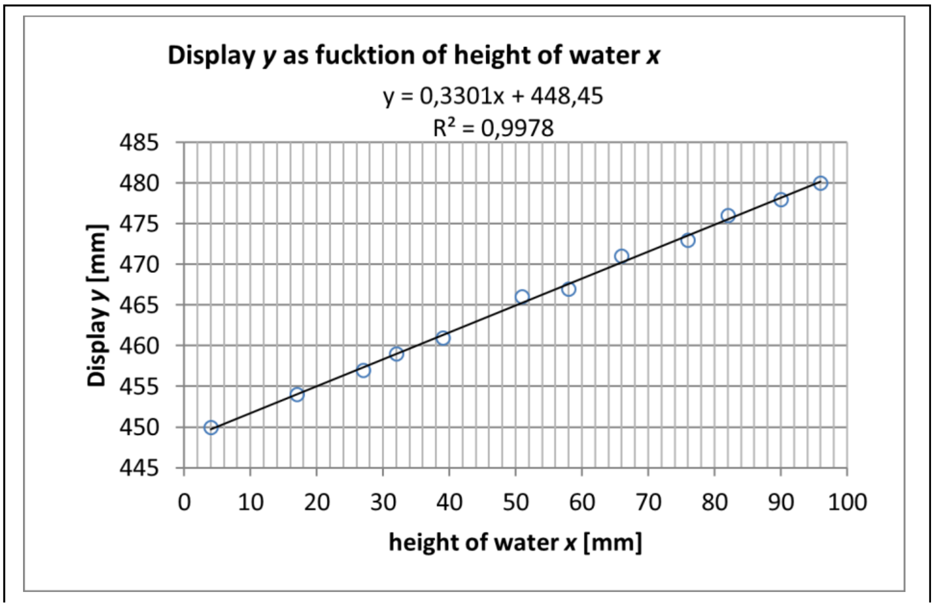
Otherwise, using min/max method

$$\Delta\theta_1 = \theta_{1\max} - \theta_1 = \cos^{-1}\left(\frac{H_{\min}}{y_{2\max} - y_{1\min}}\right) = \cos^{-1}\left(\frac{905\text{ mm}}{965\text{ mm}}\right) - \cos^{-1}\left(\frac{907\text{ mm}}{961\text{ mm}}\right) = 1.0°$$

Also, accept $\delta = 1$ mm and $\Delta\theta_1 = 0.5°$

1.4a

x	y
mm	mm
4	450
17	454
27	457
32	459
39	461
51	466
58	467
66	471
76	473
82	476
90	478
96	480



1.4b

The time it takes the light to reach the water surface is

$$t_1 = \frac{(h-x)/\cos\theta_1}{c}$$

From the water surface to the bottom the light uses the time

$$t_2 = \frac{x/\cos\theta_2}{v}$$

Total travel time forth and back

$$t = 2t_1 + 2t_2 = 2\frac{(h-x)/\cos\theta_1}{c} + 2\frac{x/\cos\theta_2}{v} = 2\frac{h-x}{c\cos\theta_1} + 2\frac{nx}{c\cos\theta_2}$$

Hence, the display will show (we simply write $n = n_{\mathrm{w}}$)

$$y = ½ct + k = \left(\frac{n}{\cos\theta_2} - \frac{1}{\cos\theta_1}\right)x + \frac{h}{\cos\theta_1} + k$$

which is a linear function of x.
Using a trigonometric identity and Snell's law,

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\frac{\sin^2\theta_1}{n^2}}$$

we get the gradient to be

$$\alpha = \frac{n}{\sqrt{1-\frac{\sin^2\theta_1}{n^2}}} - \frac{1}{\cos\theta_1} = \frac{n^2}{\sqrt{n^2-\sin^2\theta_1}} - \frac{1}{\cos\theta_1}$$

1.4c

Knowing the gradient α from the graph, we can find n solving this equation with respect to n.

Introducing a practical parameter,

$$p = \alpha + \frac{1}{\cos\theta_1}$$

our equation becomes

$$p = \frac{n^2}{\sqrt{n^2 - \sin^2\theta_1}}$$

which can be written

$$n^4 - p^2 n^2 + p^2 \sin^2\theta_1 = 0$$

and solved

$$n_{\mathrm{w}} = \sqrt{\frac{p^2 \pm \sqrt{p^4 - 4p^2 \sin^2\theta_1}}{2}} = \frac{\sqrt{2}}{2} p \sqrt{1 \pm \sqrt{1 - \left(\frac{2\sin\theta_1}{p}\right)^2}}$$

From our graph, we get $\alpha = 0.3301$. From there we find $p = 1.37865$ and hence $n_{\mathrm{w}} = 1.3437$, omitting negative solutions and solutions less than 1.

The official value of n_{w} for pure water at normal conditions is $n_{\mathrm{w}} = 1.331$ for the laser wavelength $\lambda = 635\ \mathrm{nm}$.

Just for your interest, we have the following approximations:
For small angles, we have

$$n_{\mathrm{w}} \approx \frac{\sqrt{2}}{2} p \sqrt{1 + 1 - \frac{1}{2}\left(\frac{2\sin\theta_1}{p}\right)^2} \approx p\sqrt{1 - \left(\frac{\sin\theta_1}{p}\right)^2} \approx p\left(1 - \frac{1}{2}\left(\frac{\sin\theta_1}{p}\right)^2\right)$$

For very small angles, we get

$$n_{\mathrm{w}} \approx p \approx \alpha + 1$$

It is much simpler but not recommendable to do the experiment with very small $\theta_1 \approx 0$: Reflections in the water surface will ruin the signal from the bottom.